1 Definitions

The Q-function is tail integral of a unit-Gaussian pdf, and is defined as

$$Q(z) \triangleq \int_z^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx.$$

The Q-function has the following properties:

$$\begin{aligned}
\lim_{z \to \infty} Q(z) &= 0 \\
\lim_{z \to -\infty} Q(z) &= 1 \\
Q(0) &= 1/2 \\
Q(-z) &= 1 - Q(z).
\end{aligned}$$

There are several other common notations used to denote this integral function or a close relative. The Q-function is sometime referred to as the "Gaussian Integral Function" and denoted $\mathrm{GIF}(z)$. Other functions which are closely related are the $\mathrm{erf}(\cdot)$ (error function) and $\mathrm{erfc}(\cdot)$ (complementary error function):

$$\begin{aligned}
\mathrm{erf}(z) &\triangleq \int_0^z \frac{2}{\sqrt{\pi}} e^{-x^2} dx \quad z \geq 0 \\
\mathrm{erfc}(z) &\triangleq \int_z^\infty \frac{2}{\sqrt{\pi}} e^{-x^2} dx = 1 - \mathrm{erf}(z) \quad z \geq 0
\end{aligned}$$

The Q-function is related to these functions by

$$Q(z) = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{z}{\sqrt{2}}\right)\right] = \frac{1}{2}\mathrm{erfc}\left(\frac{z}{\sqrt{2}}\right) \quad z \geq 0.$$

It is clear that if $X(u)$ is a mean zero, unit variance Gaussian random variable, that

$$Q(z) = 1 - F_{X(u)}(z).$$

A useful relation is that if $Y(u)$ is Gaussian with mean m and variance σ^2, then

$$\mathrm{Pr}\left\{Y(u) > a\right\} = Q\left(\frac{a-m}{\sigma}\right).$$

2 Numerical Computation

The Q-function must be evaluated numerically; there is no closed form solution for the integral. All numerical methods are the result of a trade-off between computational complexity and accuracy. The range for z over which the approximation is valid is also a concern. The numerical approximation which I find most useful is given by[1]

$$Q(z) \approx (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) e^{\frac{-z^2}{2}} \qquad z \geq 0,$$

where

$$\begin{aligned} t &= \frac{1}{1+Bz} & B &= 0.231641888 \\ a_1 &= 0.127414796 & a_2 &= -0.142248368 \\ a_3 &= 0.7107068705 & a_4 &= -0.7265760135 \\ a_5 &= 0.5307027145. \end{aligned}$$

The associated approximation error is guaranteed to be less than 1.5×10^{-7}. I have found that this approximation is acceptable for all practical values of z.

Another useful concept is a simple over-bound. This allows a "worst-case" scenario to be quickly evaluated. The most common overbound is

$$Q(z) \leq \frac{1}{\sqrt{2\pi} z} e^{\frac{-z^2}{2}} \qquad z > 0.$$

This bound becomes quite "tight" for large z.

The Q-function and the over-bound are plotted in Figures **??**-**??**. The plot of Figure **??** is on a log-scale to emphasize the behavior for large z.

The Q-function is tabulated in Table 1 for $z = 0$ to 10. The values of z for which $Q(z) = 10^{-k}$ for $k = 1, 2 \ldots 10$ are also given.[2]

[1]This is adapted from the erf(·) approximation of equation 7.1.26 in M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions,* Dover. Less complex approximations can also be found therein.

[2]The values in Table 1 were calculated using the approximation for $z < 4$. The values for $z \geq 4$ as well as those in the inverse Q table were taken from Albert Leon-Garcia, *Probability and Random Processes for Electrical Engineering,* Addison Wesley, 1989.

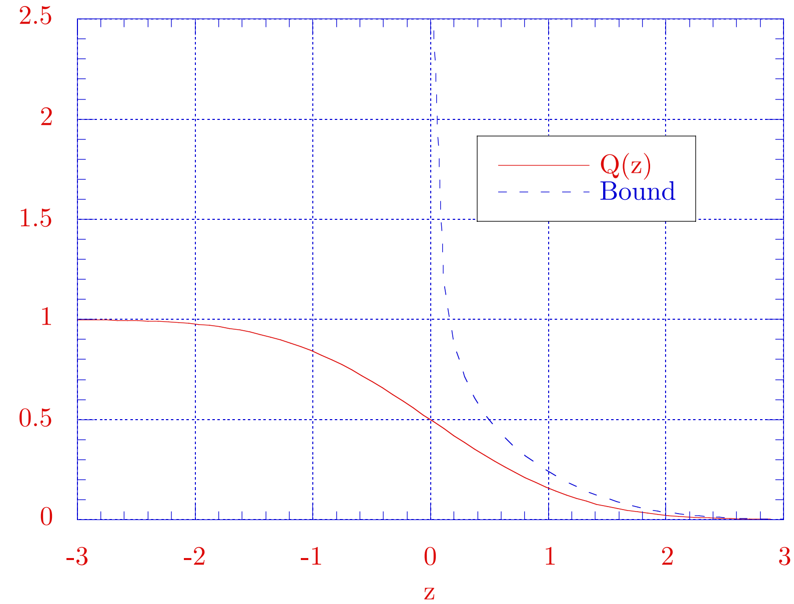

Figure 1: $Q(z)$ vs. z with linear scale.

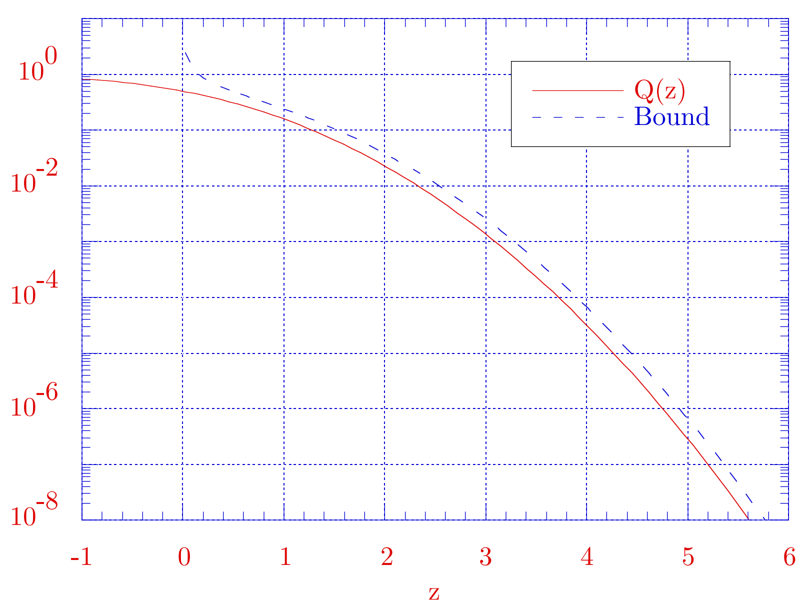

Figure 2: $Q(z)$ vs. z with log scale.

z	$Q(z)$	z	$Q(z)$
0.0	5.000e-01	3.0	1.350e-03
0.1	4.602e-01	3.1	9.677e-04
0.2	4.207e-01	3.2	6.872e-04
0.3	3.821e-01	3.3	4.835e-04
0.4	3.446e-01	3.4	3.370e-04
0.5	3.085e-01	3.5	2.327e-04
0.6	2.743e-01	3.6	1.591e-04
0.7	2.420e-01	3.7	1.078e-04
0.8	2.119e-01	3.8	7.237e-05
0.9	1.841e-01	3.9	4.812e-05
1.0	1.587e-01	4.0	3.17e-05
1.1	1.357e-01	4.5	3.40e-06
1.2	1.151e-01	5.0	2.87e-07
1.3	9.680e-02	5.5	1.90e-08
1.4	8.076e-02	6.0	9.87e-10
1.5	6.681e-02	6.5	4.02e-11
1.6	5.480e-02	7.0	1.28e-12
1.7	4.457e-02	7.5	3.19e-14
1.8	3.593e-02	8.0	6.22e-16
1.9	2.872e-02	8.5	9.48e-18
2.0	2.275e-02	9.0	1.13e-19
2.1	1.786e-02	9.5	1.05e-21
2.2	1.390e-02	10.0	7.62e-24
2.3	1.072e-02		
2.4	8.198e-03		
2.5	6.210e-03		
2.6	4.661e-03		
2.7	3.467e-03		
2.8	2.555e-03		
2.9	1.866e-03		

$Q(z)$	z
1e-01	1.2815
1e-02	2.3263
1e-03	3.0902
1e-04	3.7190
1e-05	4.2649
1e-06	4.7535
1e-07	5.1993
1e-08	5.6120
1e-09	5.9978
1e-10	6.3613

Table 1: Q-function table and inverse Q table for powers of 10.